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A Mourre Estimate and Related Bounds for Hyperbolic Manifolds with Cusps of Non-maximal Rank

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We consider the Laplace operator on quotients of hyperbolic n-dimensional space by a geometrically finite discrete group of hyperbolic isometries with parabolic subgroups of non-maximal rank. Using methods developed by the first two authors, we prove a "Mourre estimate" and commutator estimates on the Laplacian which imply absolute continuity of the spectrum and quantitative resolvent estimates. These estimates will be used elsewhere to study the scattering matrix and Eisenstein series and their meromorphic continuations, and should be useful in studying trace formulas for these discrete groups. © 1991 Academic Press, Inc.

1. INTRODUCTION

Local commutator estimates have emerged as a powerful tool for studying the essential spectrum of Schrödinger operators. Recently such

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estimates were proven for a class of elliptic operators on non-compact manifolds [4]. Included were the Laplacian on quotients of hyperbolic space by geometrically finite, discrete groups of isometries whose parabolic subgroups fixing a point at infinity were all of maximal rank. If only maximal rank cusps are present, the boundary at infinity consists of the union of a compact manifold and a finite number of points. This is no longer true for a quotient with cusps of non-maximal rank. The purpose of this paper is to prove a Mourre estimate and related bounds for the Laplacian, Δ, on such a quotient. To deal with the more complicated nature of the boundary at infinity, we work locally, defining the conjugate operator, A, as a sum of pieces, each of which reflects the local nature of the boundary at infinity in a small open set. We then prove the Mourre estimate (Theorem 2) for Δ and A. We also prove that $[\Delta, A]$ is bounded from $\mathscr{H}^2(M)$ to $L^2(M)$ and that $[[\Delta, A], A]$ is bounded from $\mathscr{H}^2$ to $\mathscr{H}^{-2}$. These estimates imply, via the Mourre theory, that any point lying above $(n-1)^2/4$ (which is not an eigenvalue) is contained in an interval, I, such that

$$\limsup_{\delta \downarrow 0} \sup_{\mu \in I} \| \langle A \rangle^{-1} (\Delta - \mu - i\delta)^{-1} \langle A \rangle^{-1} \| < \infty. \tag{1}$$

Here $\langle A \rangle = (1 + A^*A)^{1/2}$. This immediately implies that Δ has no singular continuous spectrum. Moreover, an argument in [18] shows that the bound (1) together with the related commutator bounds proven here imply that $\langle A \rangle^{-1} (\Delta - \mu - i\delta)^{-1} \langle A \rangle^{-1}$ is Hölder continuous as an element of $B(L^2(M))$ for $\mu \in I$ and $\delta \geqslant 0$.

The spectral properties of hyperbolic Laplacians have been much studied in recent years. Infinite volume quotients without cusps have been treated by Agmon [1], Mandouvalos [10–13], Mazzeo and Melrose [14], and Perry [16, 17]. For the class of manifolds considered here, absolute continuity of the spectrum, as well as non-existence of imbedded eigenvalues, was established by Lax and Phillips [6–9] (see also Phillips, Wiscott, and Woo [19]). In a forthcoming paper [5], we use the continuity of $\langle A \rangle^{-1} (\Delta - \mu - i\delta)^{-1} \langle A \rangle^{-1}$ as $\delta \downarrow 0$ in the L^2 operator norm to obtain detailed spectral information about Δ on $M = \mathbb{H}^3/\Gamma$. Combining this L^2-estimate with a resolvent localization formula, we prove asymptotic expansions for the Green's function of Δ and its derivatives as the variables approach the boundary of M at infinity. These expansions are the key ingredients in the proof of a functional relation for the Green's function on the critical line. From this functional relation, we obtain a spectral representation for Δ and define the scattering operator. The Eisenstein series and the scattering operator satisfy a functional relation on the critical line which is used to prove that the Eisenstein series have a meromorphic continuation to the entire complex plane. For an exposition of the theory of positive commutator estimates, we refer the reader to [3].

2. Description of M

Let Γ be a discrete group of orientation preserving hyperbolic isometries, and let $M = \mathbb{H}^n/\Gamma$. We assume that Γ contains no elliptic elements and is geometrically finite. Then M is a smooth n-dimensional manifold, and can be obtained from a fundamental domain for Γ by identifying points on the boundary which are equivalent under the action of Γ on $\mathbb{H}^n$. We will carefully describe the geometry of M near infinity. It is a well-known folk theorem that the essential spectrum of the Laplacian on M depends only on the behaviour of M near infinity.

We will work with the upper half space model of hyperbolic space. Thus

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}. \tag{2}$$

The metric (of constant negative curvature) is given by

$$ds^2 = x_n^{-2}(dx_1^2 + \cdots + dx_n^2), \tag{3}$$

and the invariant measure by

$$d\mu = x_n^{-n}\, dx_1 \cdots dx_n. \tag{4}$$

The (positive) Laplace operator in these coordinates is

$$\Delta = -(x_n D_n)^2 + (n-1)(x_n D_n) - \sum_{m=1}^{n-1} (x_n D_m)^2, \tag{5}$$

where D_m denotes $\partial/\partial x_m$. We write Δ in the form

$$\Delta = -B^2 + x_n^2 P + c_n^2, \tag{6}$$

where $c_n = (n-1)/2$, B is the formally antisymmetric operator,

$$B = (x_n D_n - c_n), \tag{7}$$

and P is the (Euclidean) Laplacian in the first $n-1$ coordinates,

$$P = -\sum_{m=1}^{n-1} D_m^2. \tag{8}$$

The manifold M admits a finite open cover $\{\mathscr{U}_0, \mathscr{U}_1, ..., \mathscr{U}_N\}$, where $\mathscr{U}_0$ has compact closure and each $\mathscr{U}_i$, $i = 1, ..., N$, is a neighbourhood of the boundary of M at infinity. Let $\partial_\infty \mathbb{H}^n = \mathbb{R}^{n-1} \cap \{\infty\}$ denote the boundary of $\mathbb{H}^n$ at infinity. If $\bar{\mathscr{U}}_i \cap \partial_\infty \mathbb{H}^n$ contains no points fixed by a parabolic motion in Γ, then $\mathscr{U}_i$ is a regular neighbourhood of infinity. If $\bar{\mathscr{U}}_i \cup \partial_\infty \mathbb{H}^n$ does contain a parabolic fixed point, then $\mathscr{U}_i$ is a cusp neighbourhood. We need to describe the geometry of cusp neighbourhoods in detail. The following description is culled from lecture notes of Thurston [20] and the description of Mazzeo and Phillips [15].

Let $\gamma \in \Gamma$ be parabolic with $p \in \partial_\infty \mathbb{H}^n$ its unique fixed point. Then Γ_p, the stabilizer subgroup of p in Γ, consists only of parabolic elements. This follows from the discreteness of Γ. To describe a model for a neighbourhood of p in M, let $\gamma_p \in \mathrm{Isom}(\mathbb{H}^n)$ be an element that maps p to the point at infinity and consider $\Gamma_\infty = \gamma_p \Gamma_p \gamma_p^{-1}$. Γ_∞ preserves each horosphere $\{x_n = c\}$ and acts there as a discrete subgroup of the group of Euclidean motions of $\mathbb{R}^{n-1}$ with no fixed points. Γ_∞ contains a maximal, normal abelian subgroup Γ_a of finite index in Γ_∞. The action of $\gamma \in \Gamma_a$ on $\mathbb{R}^{n-1}$ is given by

$$\gamma(x) = Ax + b,$$

where $A \in O(n-1)$ and $b \in \mathbb{R}^{n-1}$. Let $W_1(A)$ be the eigenspace of A for the eigenvalue one. Let $b = b^{\|} + b^{\perp}$ be the orthogonal decomposition of the translation part of γ relative to $W_1(A)$. Since γ has no fixed points, $b^{\|} \neq 0$ and $W_1(A)$ is non-trivial. Let $c = (A-1)^{-1} b^{\perp}$ and define the plane $E_\gamma = W_1(A) - c$. Then E_γ is γ-invariant and γ acts on E_γ by pure translations by $b^{\|}$. Since Γ_a is abelian, for any $\gamma_1, \gamma_2 \in \Gamma_a$, if b_i and c_i are the corresponding vectors defined above, then $c_1 = c_2$ and $b_1^{\|}, b_2^{\|} \in W_1(A) \cap W_2(A)$. Thus, $(\mathrm{span}\{b_1^{\|}, b_2^{\|}\} - c) \subset E_{\gamma_1} \cap E_{\gamma_2}$. In particular, the intersection is non-trivial, and γ_i acts by translation by $b_i^{\|}$ on $E_{\gamma_1} \cap E_{\gamma_2}$. Now it follows that

$$E = \bigcap_{\gamma \in \Gamma_a} E_\gamma$$

is a non-trivial hyperplane in $\mathbb{R}^{n-1}$ through c which contains a sub-hyperplane which is parallel to the span of all the $b_i^{\|}$'s. Moreover, Γ_a acts invariantly on E as a translation group of rank $k \leqslant n-1$. E is foliated by parallel planes of dimension k on which Γ_a acts by pure translations and has maximal rank. It is easy to see that the finite quotient group Γ_∞ / Γ_a permutes the leaves of this foliation, each leaf of which is isomorphic to $\mathbb{R}^k$. The foliation itself forms a Euclidean space, so Γ_∞ / Γ_a is a finite subgroup of the Euclidean group on this space and hence must have a fixed point. This means that one leaf is Γ_∞ invariant. This leaf is a k-dimensional plane in $\mathbb{R}^{n-1}$.

We have found a Γ_∞-invariant plane $\mathbb{R}^k \subset \mathbb{R}^{n-1}$ on which Γ_∞ acts as a discrete subgroup of Euclidean motions without fixed points. Let $(x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k-1}$ be coordinates corresponding to this plane. Then a general Euclidean motion γ acts on (x, z) by

$$\gamma((x, z)) = (A_1 x + A_2 z + b, B_1 x + B_2 z + c),$$

where

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} \in O(n-1),$$

$b \in \mathbb{R}^k$ and $c \in \mathbb{R}^{n-k-1}$. If $\gamma \in \Gamma_\infty$ then $c=0$ and $B_1 = 0$ because Γ_∞ acts invariantly on $\mathbb{R}^k = \{(x, z) : z = 0\}$. Then, by orthogonality, $A_2 = 0$. Hence γ acts by

$$\gamma((x, z)) = (Ax + b, Bz), \tag{9}$$

for $A \in O(k)$, $B \in O(n-k-1)$, and $b \in \mathbb{R}^k$. Let $K \subset \mathbb{R}^k$ be a fundamental domain for the action of Γ_∞ on $\mathbb{R}^k$. It follows from (9) that $K \times \mathbb{R}^{n-k-1}$ is a fundamental domain for the action of Γ_∞ on $\mathbb{R}^{n-1}$. We identify the points of the boundary of $K \times \mathbb{R}^{n-k-1}$ under the map (9). Taking $z=0$, this identifies K with a compact manifold N (which is finitely covered by a flat k-torus, according to classical Bieberbach theory) and $K \times \mathbb{R}^{n-k-1}$ with a flat vector bundle over N which we denote by V. The dimension k of the Γ_∞-invariant plane is called the rank of the cusp. When $k = n-1$, i.e., the rank is maximal, then $V = N$ is a compact manifold.

In $n=2$ dimensions, $k=1$ is the only possibility and $\mathbb{R}/\Gamma_\infty = S^1$. For $n=3$, $k=1$ or 2. When $k=1$, the transition map (9) is

$$(x, z) \to (Ax + b, z)$$

(if we consider only orientation-preserving elements) and the bundle is trivial. Non-trivial bundles occur for $n \geqslant 4$.

Finally, recall that Γ_∞ acts trivially on the x_n coordinate of $\mathbb{H}^n$. So if V is the vector bundle constructed above, $\mathbb{H}^n/\Gamma_\infty$ can be identified with $V \times \mathbb{R}_+$. This provides a model space for the cusp.

We can now give a precise description of M near infinity. Let $\{\mathscr{U}_0, \mathscr{U}_1, ..., \mathscr{U}_N\}$ be the open cover introduced above. Then each regular neighbourhood, $\mathscr{U}_i$ is isometric with the subset

$$\{(x_1, ..., x_n) : x_1^2 + \cdots + x_n^2 < 1\}$$

of the model space,

$$M_i = \mathbb{H}^n.$$

Each cusp neighbourhood, $\mathscr{U}_i$, is isometric with the subset

$$\{(x_1, ..., x_n) : x_{k+1}^2 + \cdots + x_n^2 > 1\}$$

of the model space,

$$M_i = \mathbb{H}^n/\Gamma_\infty = \mathbb{R}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+/\Gamma_\infty = V \times \mathbb{R}_+,$$

constructed from the group Γ_∞ associated with the parabolic fixed point in $\bar{\mathscr{U}}_i \cap \partial_\infty \mathbb{H}^n$.

On each model space, M_i, we have a self-adjoint realization of the Laplace operator, denoted by Δ_i, which is defined as the closure of the formal Laplacian on $C_0^\infty(M_i)$. Since $M_i = \mathbb{H}^n/\Gamma_\infty$, this operator can also be described as the operator given by (5) acting on functions on $\mathbb{H}^n$ which are Γ_∞-automorphic. The crucial point here is that Γ_∞ preserves the x_n coordinate. Thus

$$\Delta_i = -B_i^2 + x_n^2 P_i + c_n^2, \tag{10}$$

where P_i can be thought of either as the Laplacian on V, or the Euclidean Laplacian P acting on Γ_∞-automorphic functions on $\mathbb{R}^{n-1}$, where Γ_∞ acts by Euclidean motions, and B_i is the operator (7) in M_i coordinates.

We will construct a conjugate operator A_i for each model space M_i. This operator will only involve B_i, P_i, and x_{in}, all of which are invariant under the action of Γ_∞. Thus A_i can also be interpreted as the operator acting on Γ_∞-automorphic functions on $\mathbb{H}^n$, which is given formally by the same expression as the conjutate operator for $\mathbb{H}^n$.

We assume the cover has been chosen so that the intersection of two distinct cusp neighbourhoods is empty. Now suppose that two distinct neighbourhoods of infinity, $\mathscr{U}_i$ and $\mathscr{U}_j$, have non-empty intersection. Then the two coordinate systems $x_{i1}, ..., x_{in}$ and $x_{j1}, ..., x_{jn}$ are related by a hyperbolic isometry. If we write

$$\begin{aligned} x_{i1} &= x_{i1}(x_{j1}, ..., x_{jn}) \\ &\vdots \\ x_{in} &= x_{in}(x_{j1}, ..., x_{jn}) \end{aligned} \tag{11}$$

then

$$D_{im} = \sum_{l=1}^{n} \alpha_{ml}^{ij} D_{jl}, \tag{12}$$

where $\alpha_{ml}^{ij} = \partial x_{il}/\partial x_{jm}$. The functions in (11) are real analytic, so that the functions α_{kl}^{ij} and their derivatives are bounded in $\mathscr{U}_i \cap \mathscr{U}_j$. We will also need the estimates

$$0 < C_1 \leqslant x_{in}/x_{jn} \leqslant C_2 \quad \text{in} \quad \mathscr{U}_i \cap \mathscr{U}_j, \tag{13}$$

$$\frac{\partial x_{j1}}{\partial x_{in}}, ..., \frac{\partial x_{jn-1}}{\partial x_{in}} = O(x_{jn}) \quad \text{as} \quad x_{jn} \downarrow 0 \quad \text{in} \quad \mathscr{U}_i \cap \mathscr{U}_j, \tag{14}$$

and

$$\frac{x_{in}}{x_{jn}} \frac{\partial x_{jn}}{\partial x_{in}} = 1 + O(x_{jn}) \quad \text{as} \quad x_{jn} \downarrow 0 \quad \text{in} \quad \mathscr{U}_i \cap \mathscr{U}_j. \tag{15}$$

The proof of (13)–(15) for $n=3$ follows by a direct calculation using the representation of the isometry group of $\mathbb{H}^3$ by $PSL(2, \mathbb{C})$ acting on the special quaternions $\{x_1+x_2 i+x_3 j : x_1, x_2, x_3 \in \mathbb{R},\ x_3>0\}$. When $n>3$ any element of the isometry group of $\mathbb{H}^n$ can be expressed as a product of finitely many reflections in hyperbolic hyperplanes (see Ahlfors [2]). (A hyperbolic hyperplane is either a hemisphere with origin in the boundary of $\mathbb{H}^n$ at infinity, or a hyperplane perpendicular to the boundary of $\mathbb{H}^n$ at infinity.) The estimates (13)–(15) can then be proven using the explicit representations for these Möbius transformations.

To avoid cluttering the notation with unnecessary constants, we assume that $x_{in}, x_{jn} < \delta < 1$ on $\mathscr{U}_i \cap \mathscr{U}_j$. This ensures, for example, that $\log(x_{in})^{-1}$ is ounded on $\mathscr{U}_i \cap \mathscr{U}_j$.

We introduce a *partition of unity* $\{\chi_i^2\}$, $i=1, ..., N$, on M, subordinate to the cover $\{\mathscr{U}_i\}$. Each χ_i, written in local coordinates, defines a function on M_i, i.e., of the variables $x_{i1}, ..., x_{in}$. We assume that viewed as a function of these variables, χ_i is smooth up to the boundary $\{x_{in}=0\}$. Derivatives of χ_i have support for $x_{i\perp}=(x_{i1}, ..., x_{i(n-1)})$ in a compact region, and for $x_{in} < \delta < 1$.

We also need *identification operators*, J_i, for $i=1, ..., N$. These identify a function on M with support in $\mathscr{U}_i$ with a function on M_i. When dealing with local, i.e., differential, operators, this just amounts to writing functions in local coordinates, and we will sometimes suppress this notation. However our conjugate operator will be a sum of non-local pieces and it will be important to keep track on which space each piece is acting.

We will make use of the *scale of spaces* $\mathscr{H}^s(M)$, associated with the Laplace operator Δ acting in $L^2(M)$. We will also use the spaces $\mathscr{H}^2(M_i)$ associated with the model spaces and their Laplacians. These are defined in [3].

The proof of the following lemma is elementary.

LEMMA 2.1. *The following operators are bounded for all $s \in \mathbb{R}$:*

1. *$J_i\chi_i$ and $\chi_i J_i$ from $\mathscr{H}^s(M)$ to $\mathscr{H}^s(M_i)$,*
2. *$J_i^*\chi_i$ and $\chi_i J_i^*$ from $\mathscr{H}^s(M_i)$ to $\mathscr{H}^s(M)$,*
3. *χ_i from $\mathscr{H}^s(M)$ to $\mathscr{H}^s(M)$ and from $\mathscr{H}^s(M_i)$ to $\mathscr{H}^s(M_i)$.*

3. DEFINITION OF THE CONJUGATE OPERATOR

We begin by defining model conjugate operators, acting in $L^2(M_i)$. Define

$$L_i = \log(x_{in}^2 P_i) = 2\log(x_{in}) + \log(P_i), \tag{16}$$

where P_i is the Laplacian defined in the previous section. Then we define

$$A_i = \xi(L_i/S)(L_i - 2S)B_i + B_i(L_i - 2S)\xi(L_i/S), \tag{17}$$

where $\xi(x)$ is a smooth cutoff function which vanishes for $x > 1$ and is identically one for $x < 1/2$, and B_i is the operator (7) in M_i coordinates. Here S is a real parameter. Using the commutation relation

$$[B_i, F(L_i)] = 2F'(L_i), \tag{18}$$

we can also write A_i as

$$A_i = 2\xi(L_i/S)(L_i - 2S)B_i + 2F_1(L_i/S), \tag{19}$$

where

$$F_1(x) = \xi(x) + \xi'(x)(x-2). \tag{20}$$

It is straightforward to calculate the model commutators, $[\Delta_i, A_i]$. We obtain

$$[\Delta_i, A_i] = -8B_iF_1(L_i/S)B_i + F_2(L_i/S), \tag{21}$$

where F_1 is given by (20) and F_2 by

$$\begin{aligned} F_2(x) = {} & -8\xi'''(x)(x-2)/S^2 - 24\xi''(x)/S^2 \\ & + 4\xi(x)S(-x+2)\exp(Sx). \end{aligned} \tag{22}$$

We now can define the conjugate operator acting in $L^2(M)$. It is given by

$$A = \sum_{i=1}^{N} \chi_i J_i^* A_i J_i \chi_i. \tag{23}$$

4. Model Space Estimates

In this section we prove the Mourre estimate and commutator bounds for the model operators acting in $L^2(M_i)$. We will require the following bounds:

Lemma 4.1. *The following operators are bounded*:

(1) B_i *from* $\mathscr{H}^s(M_i)$ *to* $\mathscr{H}^{s-1}(M_i)$, *for* $s \in [-1, 2]$,

(2) $x_{in}D_{ij}$ *for* $1 \leqslant j \leqslant n-1$ *from* $\mathscr{H}^s(M_i)$ *to* $\mathscr{H}^{s-1}(M_i)$, *for* $s \in [-1, 2]$,

(3) *$F[L_i)$ from $\mathscr{H}^s(M_i)$ to $\mathscr{H}^s(M_i)$, for $s \in [-2, 2]$, where F and its first 2 derivatives are bounded functions,*

(4) *$\xi(L_i/S)L_i(\log(x_{in}))^{-1}\eta(x_{i\perp}, x_{in})$ from $\mathscr{H}^s(M_i)$ to $\mathscr{H}^s(M_i)$, where η has support for $x_{i\perp}$ in a fixed compact set and for $x_{in} < \delta < 1$, and $s \in [-2, 2]$.*

Proof. The proofs of (1) and (2) are elementary. Since F is bounded, $F(L_i)$ is clearly a bounded operator in L^2. Recall that F is bounded from $\mathscr{H}^s$ to $\mathscr{H}^s$ whenever the operator $(\Delta_i + 1)^{s/2}F(\Delta_i + 1)^{-s/2}$ is L^2 bounded. When $s = 2$,

$$\begin{aligned}(\Delta_i + 1)^{s/2} & F(L_i)(\Delta_i + 1)^{-s/2} \\ &= F(L_i) + [B_i^2, F(L_i)](\Delta_i + 1)^{-1} \\ &= F(L_i) + (4F''(L_i) + 4F'(L_i)B_i)(\Delta_i + 1)^{-1},\end{aligned} \tag{24}$$

which is bounded by part (1). Similarly for $s = -2$, and so for all $s \in \mathbb{C}$ with $\Re(s) \in \{-2, 2\}$, the right side of (24) is bounded. It now follows by complex interpolation that the right side of (24) is bounded for all s in the strip $-2 \leqslant \Re(s) \leqslant 2$. In particular, (3) holds.

The proof of (4) is more subtle. We begin by proving L^2 boundedness. Using a partition of unity we write $\log(x) = \log_-(x) + \log_+(x)$ where $\log_-(x)$ has support for $x < 2$ and $\log_+(x)$ has support for $x > 1$. Then

$$\begin{aligned}\xi(L/S)&L(\log(x_n))^{-1}\eta(x_\perp, x_n) \\ &= \xi(L/S)(2\log(x_n) + \log_-(P) \\ &\quad + \log_+(P))(\log(x_n))^{-1}\eta(x_\perp, x_n).\end{aligned} \tag{25}$$

Here we have dropped the i subscript. The first term on the right is obviously bounded. To see that the $\log_+$ term is bounded, notice that on the support of $\eta\xi$, $\log(P) \leqslant 2\,|\log(x_n)| + S$. Thus, on this support,

$$0 \leqslant \xi(L/S)\log_+(P) \leqslant \xi(L/S)(2\,|\log(x_n)| + S), \tag{26}$$

which implies that the $\log_+$ term is bounded. To bound the $\log_-$ term it is sufficient, given (3), to show that $\log_-(P)\eta$ is bounded. Since

$$\|\log_-(P)\eta\psi\|^2 = \int_{\mathbb{R}_+} \|\log_-(P)\eta(x_\perp, x_n)\psi(x_\perp, x_n)\|^2_{L^2(\mathbb{R}^{n-1}, dx_\perp)} x_n^{-n}\, dx_n, \tag{27}$$

it suffices to show that $\log_-(P)\eta$ is bounded on $L^2(\mathbb{R}^{n-1}, dx_\perp)$ with bound uniform in x_n. But

$$
\begin{aligned}
\|\log_-(P)\eta(x_\perp, x_n)\psi\|_{L^2(dx_\perp)} &= \|\log_-(k^2)\widehat{\eta\psi}\|_{L^2(d^{n-1}k)} \\
&\leqslant \|\log_-\|_2\, \|\widehat{\eta\psi}\|_\infty \\
&\leqslant \|\log\quad\|_2\, \|\eta\psi\|_1 \\
&\leqslant \|\log_-\|_2\, \|\eta\|_2\, \|\psi\|_2, \qquad (28)
\end{aligned}
$$

with $\|\eta\|_2$ uniformly bounded in x_n. This proves the L^2 boundedness. The $\mathscr{H}^s$ boundedness can now be proven by the method outlined above. ∎

LEMMA 4.2. *$[\Delta_i, A_i]$ is bounded from $\mathscr{H}^2(M_i)$ to $L^2(M_i)$.*

Proof. We rewrite the commutator as

$$[\Delta_i, A_i] = -8F_1(L_i/S)(B_i)^2 + F_3(L_i/S)B_i + F_2(L_i/S), \qquad (29)$$

where

$$F_3(x) = -16F_1'(x)/S, \qquad (30)$$

and apply the bounds in Lemma 4.1. ∎

LEMMA 4.3 (Model Mourre Estimate). *Let $c_n = (n-1)/2$. For every $\lambda > c_n^2$ and every $\varepsilon > 0$ there is a smoothed characteristic function, f of some interval about λ such that for large enough S,*

$$f(\Delta_i)[\Delta_i, A_i]\, f(\Delta_i) \geqslant 8(\lambda - c_n^2 - \varepsilon)\, f^2(\Delta_i). \qquad (31)$$

Proof. We begin with expression (21) for the commutator. Note that $\xi'(x)(x-2) \geqslant 0$, so $F_1 \geqslant \xi$. Thus

$$[\Delta_i, A_i] \geqslant -8B_i\xi(L_i/S)B_i + F_4(L_i/S) + 4\xi(L_i/S)(-L_i + 2S)(x_{in})^2 P_i, \quad (32)$$

where

$$F_4(x) = -8S^{-2}\xi'''(x)(x-2) - 24S^{-2}\xi''(x). \qquad (33)$$

Now $-L_i + 2S \geqslant S$ on the support of ξ. Choose $S > 2$. Then, since $F_4 = O(S^{-2})$,

$$
\begin{aligned}
[\Delta_i, A_i] \geqslant 8(\Delta_i - 1) &- 8B_i(1-\xi)B_i \\
&- 8(1-\xi)(x_{in})^2 P_i + O(S^{-2}). \qquad (34)
\end{aligned}
$$

Now we multiply this inequality from both sides by a smoothed characteristic function, $f(\Delta_i)$. We wish to show that the Mourre estimate holds as we shrink the support of f and let S tend to infinity. This follows from

$$\lim_{S\to\infty} \limsup_{a\to\infty} \|f(a(\Delta_i - \lambda))(1 - \xi(L_i/S))\| = 0. \qquad (35)$$

But $f(x)^2 \leqslant |x-z|^{-2}$ for some complex number z. Thus

$$\begin{aligned}\|f(1-\xi)\|^2 &= \|(1-\xi) f^2(1-\xi)\| \\ &\leqslant \|(1-\xi)\, |a(\Delta_i-\lambda)-z|^{-2}\,(1-\xi)\| \\ &\leqslant \|(1-\xi)(a(\Delta_i-\lambda)-z)^{-1}\|^2. \end{aligned} \tag{36}$$

Therefore it suffices to show that

$$\lim_{S\to\infty} \limsup_{a\to\infty} \|(1-\xi)(a(\Delta_i-\lambda)-z)^{-1}\| = 0. \tag{37}$$

To prove (37) begin with

$$\begin{aligned}&(1-\xi)^2\,(a(\Delta_i-\lambda)-z)+(a(\Delta_i-\lambda)-\bar{z})(1-\xi)^2 \\ &\quad = 2a(1-\xi)\left(-B^2+x_n^2P+1-\lambda-\frac{\Re(z)}{a}\right) \\ &\qquad \times(1-\xi)+aO(S^{-2}) \geqslant 2a(1-\xi)^2+aO(S^{-2}). \end{aligned} \tag{38}$$

Here we used the fact that, for large S, x_n^2P is large on the support of $(1-\xi)$. If we multiply this equation from the left with $(a(\Delta_i-\lambda)-z)^{-1}$ and from the right with $(a(\Delta_i-\lambda)-\bar{z})^{-1}$ and take norms, we get

$$\begin{aligned}&2a\,\|(1-\xi)(a(\Delta_i-\lambda)-z)^{-1}\|^2 \\ &\quad \leqslant \|(1-\xi)^2\,(a(\Delta_i-\lambda)-z)^{-1}\|+aO(S^{-2}), \end{aligned} \tag{39}$$

which, in view of the estimate $\|(1-\xi)\| \leqslant 1$, proves (37). ∎

5. Commutator Bound

Theorem 5.1. *$[\Delta, A]$ is bounded from $\mathscr{H}^2(M)$ to $L^2(M)$.*

Proof. From the definition (23) of A it follows that

$$\begin{aligned}[\Delta, A] &= \left[\Delta, \sum_{i=1}^{N} J_i^* \chi_i A_i \chi_i J_i\right] \\ &= \sum_{i=1}^{N} J_i^*[\Delta_i, \chi_i A_i \chi_i] J_i. \end{aligned} \tag{40}$$

Therefore, it suffices to show that $[\Delta_i, \chi_i A_i \chi_i]$ is bounded from $\mathscr{H}^2(M_i)$ to $L^2(M_i)$. We compute

$$[\Delta_i, \chi_i A_i \chi_i] = [\Delta_i, \chi_i] A_i \chi_i + \chi_i[\Delta_i, A_i]\chi_i + \chi_i A_i[\Delta_i, \chi_i]. \tag{41}$$

This middle term is bounded, by Lemma 4.2. Now consider the last term of (41). A calculation shows

$$[\Delta_i, \chi_i] = -2x_n \frac{\partial \chi}{\partial x_n} B - x_n \frac{\partial}{\partial x_n}\left(x_n \frac{\partial \chi}{\partial x_n}\right) \tag{42}$$

$$-2\sum_{m=1}^{n-1}\left(x_n \frac{\partial \chi}{\partial x_m} x_n D_m + x_n^2 \frac{\partial^2 \chi}{\partial x_m^2}\right). \tag{43}$$

Here we dropped the i subscripts. By Lemma 4.1 this is bounded from $\mathscr{H}^2$ to $\mathscr{H}^1$. Moreover, this is a local operator wth support for $x_\perp$ in a fixed compact region and for $x_n < 1$. Thus we can introduce a cutoff function, η, with these support properties to the left of $[\Delta_i, \chi_i]$ in the last term of (41). So to prove that that term is bounded, it suffices to show that $\chi_i A_i x_{in} \eta$ is bounded from $\mathscr{H}^1$ to L^2. This follows from expression (19) for A_i and Lemma 4.1(3), (4). Thus the last term in (41) is bounded. The first term is handled similarly. ∎

6. The Mourre Estimate

To prove the Mourre estimate, we will require the following lemma. Let C_∞ denote the sup norm closure of C_0^∞.

Lemma 6.1. *Let $f \in C_\infty$. Then the following operators are compact*:

(1) $f(\Delta)\eta$ *where* $\eta \in C_\infty$,

(2) $[\chi_i, f(\Delta_i)]$,

(3) $(f(\Delta)J_i^* - J_i^* f(\Delta_i))\chi_i$.

Proof. It suffices to prove this lemma for $f(x) = (x-z)^{-1}$ where z is in the resolvent set of Δ and Δ_i. It then follows from a Stone–Weierstrass argument (see, for example, [3]) that the lemma holds for $f \in C_\infty$. Parts (1) and (2) now follow from standard elliptic theory and a Sobolev imbedding theorem.

To prove (3) we begin with the equation

$$(\Delta - z)J_i^* \chi_i = J_i^*(\Delta_i - z)\chi_i, \tag{44}$$

and multiply from the left with $(\Delta - z)^{-1}$ and from the right with $(\Delta_i - z)^{-1}$. This leads to the equation

$$\begin{aligned}((\Delta - z)^{-1} J_i^* - J_i^*(\Delta_i - z)^{-1})\chi_i \\ = J_i^*(\Delta_i - z)^{-1} [\Delta_i, \chi_i](\Delta_i - z)^{-1} \\ - (\Delta - z)^{-1} J_i^* [\Delta_i, \chi_i](\Delta_i - z)^{-1},\end{aligned} \tag{45}$$

whose left side is compact. ∎

THEOREM 6.2 (The Mourre Estimate). *Let $c_n = (n-1)/2$. For every $\lambda > c_n^2$ and every $\varepsilon > 0$ there is a smoothed characteristic function, f, of some interval about λ and a compact operator, K, such that for large enough S,*

$$f(\Delta)[\Delta, A]f(\Delta) \geqslant 8(\lambda - c_n^2 - \varepsilon) f^2(\Delta) + K. \tag{46}$$

Proof. We begin with the equation

$$\begin{aligned} f(\Delta)J_i^*[\Delta_i, \chi_i A_i \chi_i]J_j f(\Delta) &= f(\Delta)J_i^* \chi_i A_i [\Delta_i, \chi_i]J_j f(\Delta) \\ &\quad + f(\Delta)J_i^*[\Delta_i, \chi_i]A_i \chi_i J_i f(\Delta) \\ &\quad + f(\Delta)J_i^* \chi_i [\Delta_i, A_i]\chi_i J_i f(\Delta). \end{aligned} \tag{47}$$

Let $\tilde{\chi}$ be a function which equals one on the support of $\nabla\chi_i$ and whose support is only slightly bigger. The first term can be rewritten as

$$(f(\Delta)x_n \log(x_n)\tilde{\chi}J_i^*)(\log(x_n)^{-1} x_n^{-1} \chi_i A_i [\Delta_i, \chi_i]J_j f(\Delta)). \tag{48}$$

Here $x_n = x_{in}$ is well defined on the support of $\tilde{\chi}$. The first factor is compact by Lemma 6.1 while the second factor is bounded, by an argument similar to the one in Lemma 4.1. Thus this term is compact. A similar argument shows that the second term on the right of (47) is compact. The last term can be written

$$\begin{aligned} &f(\Delta)J_i^* \chi_i [\Delta_i, A_i]\chi_i J_i f(\Delta) \\ &\quad = J_i^* f(\Delta_i)\chi_i [\Delta_i, A_i]\chi_i f(\Delta_i)J_i + K \\ &\quad = J_i^* \chi_i f(\Delta_i)[\Delta_i, A_i] f(\Delta_i)\chi_i J_i + K \\ &\quad \geqslant 8(\lambda - c_n^2 - \varepsilon)J_i^* \chi_i f(\Delta_i)^2 \chi_i J_i + K \\ &\quad = 8(\lambda - c_n^2 - \varepsilon) f(\Delta)\chi_i^2 f(\Delta) + K. \end{aligned} \tag{49}$$

Here K denotes a compact operator. In the first and last lines we used part (3) of Lemma 6.1. In the second and last lines we used part (2). In the third line we used the model space Mourre estimate. Here we assume the suport of f is sufficiently small about λ and that S is sufficiently large. Combining the estimates above, we obtain

$$\begin{aligned} f(\Delta)[\Delta, A]f(\Delta) &= \sum_{i=1}^{N} f(\Delta)J_i^*[\Delta_i, \chi_i A_i \chi_i]J_i f(\Delta) \\ &\geqslant \sum_{i=1}^{N} 8(\lambda - c_n^2 - \varepsilon) f(\Delta)\chi_i^2 f(\Delta) + K, \\ &= 8(\lambda - c_n^2 - \varepsilon) f(\Delta)(1 - \chi_0^2) f(\Delta) + K, \\ &= 8(\lambda - c_n^2 - \varepsilon) f(\Delta)^2 + K. \end{aligned} \tag{50}$$

This completes the proof. ∎

7. Double Commutator Bound

Lemma 7.1. *Let χ be supported on $\mathscr{U}_i \cap \mathscr{U}_j$. Then*

$$(x_{in}^2 P_i - x_{jn}^2 P_j)\chi = E x_{in} \chi, \tag{51}$$

where E is $\mathscr{H}^1$ to $\mathscr{H}^{-1}$ bounded.

Proof. Since the Laplacian is the same when written in x_i or x_j coordinates,

$$x_{in}^2 P_i - x_{jn}^2 P_j = B_j^2 - B_i^2, \tag{52}$$

on the support of χ. Now

$$D_{in} = E_1 + \frac{\partial x_{jn}}{\partial x_{in}} D_{jn}, \tag{53}$$

where

$$E_1 = \sum_{k=1}^{n-1} \frac{\partial x_{jk}}{\partial x_{in}} D_{jk}. \tag{54}$$

The estimate (14) implies that E_1 is bounded from $\mathscr{H}^1$ to L^2. Thus

$$x_{in} D_{in} = E_1 x_{in} + \frac{x_{in}}{x_{jn}} \frac{\partial x_{jn}}{\partial x_{in}} x_{jn} D_{jn}. \tag{55}$$

Applying the estimate (15) we therefore find that

$$B_i = B_j + E_2 x_{in}, \tag{56}$$

where E_2 is bounded from $\mathscr{H}^1$ to L^2. To complete the proof, we observe that

$$B_i^2 - B_j^2 = (B_i + B_j)(B_i - B_j) + [B_i, B_j]. \quad \blacksquare \tag{57}$$

Theorem 7.2. *The double commutator, $[[\Delta, A], A]$, is bounded from $\mathscr{H}^2(M)$ to $\mathscr{H}^{-2}(M)$.*

Proof. The double commutator is given by

$$[[\Delta, A], A] = \sum_{i=1}^{N} \sum_{j=1}^{N} [[\Delta, \chi_i J_i^* A_i J_i \chi_i], \chi_j J_j^* A_j J_j \chi_j], \tag{58}$$

so it suffices to bound each term on the right. Now

$$\begin{aligned}
&[[\Delta, \chi_i J_i^* A_i J_i \chi_i], \chi_j J_j^* A_j J_j \chi_j] \\
&\quad = [[\Delta, \chi_i] J_i^* A_i J_i \chi_i, \chi_j J_j^* A_j J_j \chi_j] \\
&\qquad + [\chi_i J_i^* [\Delta_i, A_i] J_i \chi_i, \chi_j J_j^* A_j J_j \chi_j] \\
&\qquad + [\chi_i J_i^* A_i J_i [\Delta, \chi_i], \chi_j J_j^* A_j J_j \chi_j].
\end{aligned} \tag{59}$$

To handle the first and last terms of (59) we expand the outer commutator. This produces four terms, a typical one being

$$[\Delta, \chi_i] J_i^* A_i J_i \chi_i \chi_j J_j^* A_j J_j \chi_j. \tag{60}$$

Since $[\Delta, \chi_i]$ has support for $x_{i\perp}$ in a compact set and for $x_{in} < \delta < 1$, we can insert a function, η, with the same support properties, but which equals one on the support of $[\Delta, \chi_i]$. Then we can rewrite (60) as

$$\begin{aligned} &[\Delta, \chi_i] \log(x_{in})^2 J_i^* \log(x_{in})^{-1} \eta A_i J_i J_j^* \log(x_{in})^{-1} \chi_i \chi_j A_j J_j \chi_j \\ &\quad + [\Delta, \chi_i] \log(x_{in})^2 J_i^* \log(x_{in})^{-1} \eta \\ &\quad \times [\log(x_{in})^{-1}, A_i] J_j^* \chi_i \chi_j A_j J_j \chi_j. \end{aligned} \tag{61}$$

It follows from Lemma 4.1 that $\log(x_{in})^{-1} \chi_i \chi_j A_j$ is bounded from $\mathscr{H}^2$ to $\mathscr{H}^1$, that $\log(x_{in})^{-1} \eta A_i$ is bounded from $\mathscr{H}^1$ to L^2 and that $[\Delta, \chi_i] \log(x_{in})^2$ is bounded from L^2 to $\mathscr{H}^{-1}$. Thus the first term in (61) is bounded from $\mathscr{H}^2$ to $\mathscr{H}^{-1}$. The second term can be handled similarly.

We now expand the middle term of (59). We use (21). This gives rise to 2 similar terms, corresponding to the two terms on the right side of (21). We will show how to bound the first of these, namely

$$[\chi_i J_i^* B_i F_1(L_i/S) B_i J_i \chi_i, \chi_j J_j^* A_j J_j \chi_j]. \tag{62}$$

We dropped a factor of -8. Since the functions χ_i and χ_j always appear together, we can assume that they are both supported on a set which is only slightly larger than the intersection of the supports. After inserting a function $\tilde{\chi}$ which has roughly the same support properties, but equals one on the support of these modified functions, and some computation, we arrive at

$$\begin{aligned} &[\chi_i J_i^* B_i F_1(L_i/S) B_i J_i \chi_i, \chi_j J_j^* A_j J_j \chi_j] \\ &\quad = \chi_i B_i \tilde{\chi} J_i^* F_1 J_i \tilde{\chi} [B_i \chi_i, R_j] \\ &\quad\quad + \chi_i B_i [\tilde{\chi} J_i^* F_1 J_i \tilde{\chi}, R_j] B_i \chi_i \\ &\quad\quad + [\chi_i B_i, R_j] \tilde{\chi} J_i^* F_1 J_i \tilde{\chi} B_i \chi_i, \end{aligned} \tag{63}$$

where

$$R_j = \chi_j J_j^* A_j J_j \chi_j. \tag{64}$$

Thus we see it suffices to bound

$$[B_i \chi_i, \chi_j A_j \chi_j], \tag{65}$$

$$[\chi_i B_i, \chi_j A_j \chi_j], \tag{66}$$

from $\mathscr{H}^2$ to L^2, and

$$[\tilde{\chi} J_i^* F_1 J_i \tilde{\chi}, \chi_j J_j^* A_j J_j \chi_j] \tag{67}$$

from $\mathscr{H}^1$ to $\mathscr{H}^{-1}$.

We begin with (65). We write this as three terms,

$$[B_i \chi_i, \chi_j] A_j \chi_j + \chi_j [B_i \chi_i, A_j] \chi_j + \chi_j A_j [B_i \chi_i, \chi_j]. \tag{68}$$

It is straightforward to bound the first and last of these terms. To handle the middle term, we need to change from $\mathscr{U}_i$ to $\mathscr{U}_j$ variables. We have

$$B_i \chi_i = \sum_{m=1}^{n} \beta_{nm} x_{jn} D_{jm} - \beta, \tag{69}$$

where

$$\beta_{nm} = (x_{in}/x_{jn}) \alpha_{nm}^{ij} \chi_i, \tag{70}$$

and

$$\beta = [B_i, \chi_i] + \chi_i = x_{in} \frac{\partial \chi_i}{\partial x_{in}} + \chi_i. \tag{71}$$

The functions β_{nm} and β have compact $x_\perp$ support. We use (19) to write A_j. Neglecting obviously bounded terms, this leaves

$$\sum_{m=1}^{n} 2\chi_j [\beta_{nm} x_{jn} D_{jm}, \xi L_j B_j] \chi_j - 2\chi_j [\beta, \xi L_j B_j] \chi_j, \tag{72}$$

which we must show is bounded from $\mathscr{H}^2$ to L^2. Since $x_{jn} D_{jm}$, for $m < n$, commutes with ξL_j it is straightforward to do this, using Lemma 4.1, given the L^2 boundedness of

$$[\tilde{\beta}, \xi L_j], \tag{73}$$

where $\tilde{\beta}$ is one of the functions β, β_{nm}.

Before proving this bound we consider the terms (66) and (67). The (66) term is handled similarly to the term just considered. To bound (67) we begin with the observation that the commutator is bounded if we replace F_1 by 1. This follows from (73). Thus we can replace F_1 with $F_1 - 1$ which has support for $x_{in}^2 P_i$ bounded away from zero. Now write A_j using (19). The main term is

$$\begin{aligned} &[\tilde{\chi} J_i^* (F_1 - 1) J_i \tilde{\chi}, \chi_j J_j^* \xi L_j J_j \chi_j] B_j \\ &\quad = [\tilde{\chi} J_i^* (F_1 - 1)(x_{in}^2 P_i)^{-1} (x_{in}^2 P_i - x_{jn}^2 P_j + x_{jn}^2 P_j) \\ &\qquad \times J_j \tilde{\chi}, \chi_j J_j^* \xi L_j J_j \chi_j] B_j. \end{aligned} \tag{74}$$

Now $(F_1 - 1)(x_{in}^2 P_i)^{-1}$ is bounded. We need the fact, proven in the lemma above, that

$$x_{in}^2 P_i - x_{jn}^2 P_j = E x_{in}, \tag{75}$$

where E is $\mathscr{H}^1$ to $\mathscr{H}^{-1}$ bounded. Ths gives us a factor of x_{in} which we can move against the L_j singularity to bound it. The remaining term has a factor of $x_{jn}^2 P_j$. Since $x_{jn}^2 P_j \xi L_j$ is bounded, this term can be bounded by moving $x_{jn}^2 P_j$ against the ξL_j term and estimating the resulting commutator terms.

To complete the proof, it remains to bound (73). Dropping the tilde and the j subscript, we have

$$[\beta, \xi L] = [\beta, \xi \cdot (2 \log(x_n) + \log_-(P) + \log_+(P))], \tag{76}$$

where $\log_-$ and $\log_+$ are as in the proof of Lemma 4.1. The $\log_-$ term is bounded by the estimate (28). The other two terms both have the form

$$[\beta, p(D_\perp; x_n)], \tag{77}$$

where $p(k; x_n)$ equals either $2\xi(\log(x_n^2 k^2)) \log(x_n)$ or $\xi(\log(x_n^2 k^2)) \log_+(k^2)$. As in the proof of Lemma 4.1, part (4), we can view the commutators as operators in $L^2(\mathbb{R}^{n-1})$ that depend parametrically on x_n, and it suffices to show that they are bounded on $L^2(\mathbb{R}^{n-1})$ uniformly in $0 < x_n < 1$. Since $p(D_\perp; x_n)$ is a Fourier multiplier, it is not difficult to see that

$$\begin{aligned}\|[\beta, p]\psi\| &= \left\| \int_{\mathbb{R}^{n-1}} (\beta(x_\perp, x_3) - \beta(y_\perp, x_3)) \right.\\ &\qquad \left. \times \hat{p}(x_\perp - y_\perp; x_3)\psi(x_\perp)\, dx_\perp \right\|_{L^2(dt_\perp)} \\ &= \left\| \int_{\mathbb{R}^{n-1}} \frac{(\beta(x_\perp, x_n) - \beta(y_\perp, x_n))}{|x_\perp - y_\perp|^2} (x_\perp - y_\perp) \right. \\ &\qquad \left. \cdot \widehat{\nabla p}(x_\perp - y_\perp; x_n)\psi(x_\perp)\, dx_\perp \right\|_{L^2(dy_\perp)}. \end{aligned} \tag{78}$$

Let F denote the smooth vector-valued function

$$F(x_\perp, y_\perp; x_n) = \frac{(\beta(x_\perp, x_n) - \beta(y_\perp, x_n))}{|x_\perp - y_\perp|^2} (x_\perp - y_\perp). \tag{79}$$

Then, by the Plancherel theorem, the right side of (78) equals

$$\begin{aligned}\|\nabla_k p(k; x_n) \cdot \widehat{F\psi}\|_{L^2(dk)} &\leqslant \|\nabla p\|_\infty \|F\psi\|_2 \\ &\leqslant \|\nabla p\|_\infty \|F\|_\infty \|\psi\|_2. \end{aligned} \tag{80}$$

Therefore, to complete the proof, we must establish the L^∞ bound on ∇p. When $p = \xi(\log(x_n^2 k^2)) \log_+(k^2)$, this is a simple calculation. When $p = 2\xi(\log(x_n^2 k^2)) \log(x_n)$,

$$|\nabla_k p| = \left| 4\xi'(\log(x_n^2 k^2)) \frac{\log(x_n)}{|k|} \right|$$

$$= \left| 4\xi'(\log(x_n^2 k^2)) \frac{x_n \log(x_n)}{x_n\,|k|} \right|.$$

This is bounded because $x_n \log(x_n)$ is bounded for small x_n, and $x_n\,|k|$ is bounded below on the support of ξ'. This completes the proof. ∎

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